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Note

Signed star domatic number of a graph

M. Atapour [a], S.M. Sheikholeslami [a,*], A.N. Ghameshlou [b], L. Volkmann [c]

[a] Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, Islamic Republic of Iran

[b] Department of Mathematics, University of Mazandaran, Babolsar, Islamic Republic of Iran

[c] Lehrstuhl II für Mathematik, RWTH-Aachen University, 52056 Aachen, Germany

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ABSTRACT

Let G be a simple graph without isolated vertices with vertex set $V(G)$ and edge set $E(G)$. A function $f : E(G) \longrightarrow \{-1, 1\}$ is said to be a signed star dominating function on G if $\sum_{e\in E(v)} f(e) \geq 1$ for every vertex v of G, where $E(v) = \{uv \in E(G) \mid u \in N(v)\}$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed star dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq 1$ for each $e \in E(G)$, is called a signed star dominating family (of functions) on G. The maximum number of functions in a signed star dominating family on G is the signed star domatic number of G, denoted by $d_{SS}(G)$.

In this paper we study the properties of the signed star domatic number $d_{SS}(G)$. In particular, we determine the signed domatic number of some classes of graphs.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [1,6] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. For every non-empty subset E' of $E(G)$, the subgraph $G[E']$ induced by E' is the graph whose vertex set consists of those vertices of G incident with at least one edge of E' and whose edge set is E'.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e\in S} f(e)$. The *edge neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges incident with the vertex v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e\in E_G(v)} f(e)$. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed star dominating function* (SSDF) on G, if $f(v) \geq 1$ for every vertex v of G. The *signed star domination number* of a graph G is $\gamma_{SS}(G) = \min\{\sum_{e\in E} f(e) \mid f$ is a SSDF on $G\}$. The signed star dominating function f on G with $f(E(G)) = \gamma_{SS}(G)$ is called a *$\gamma_{SS}(G)$-function*. The signed star domination number was introduced by Xu in [7] and has been studied by several authors (see for instance [2,5,8]).

A set $\{f_1, f_2, \ldots, f_d\}$ of signed star dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq 1$ for each $e \in E(G)$, is called a *signed star dominating family* (of functions) on G. The maximum number of functions in a signed star dominating family on G is the *signed star domatic number* of G, denoted by $d_{SS}(G)$.

In this paper we first study the basic properties of $d_{SS}(G)$ in which some of them are analogous to those of the signed domatic number $d_S(G)$ in [3,4], and then we determine the signed star domatic number of some classes of graphs.

* Corresponding author. Tel.: +98 4112364236; fax: +98 4124327541.
E-mail addresses: s.m.sheikholeslami@azaruniv.edu, sm_sheikholeslami@yahoo.com (S.M. Sheikholeslami), volkm@math2.rwth-aachen.de (L. Volkmann).

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Our first proposition shows that the signed star domatic number $d_{SS}(G)$ is well defined for every graph G.

Proposition 1. *The signed star domatic number $d_{SS}(G)$ is well defined for each graph G.*

Proof. Since the function $f : E(G) \to \{-1, 1\}$ with $f(e) = 1$ for each $e \in E(G)$ is a signed star dominating function on G, the family $\{f\}$ is a signed star dominating family on G. Therefore, the set of signed star dominating functions on G is non-empty and there exists the maximum of their cardinalities, which is the signed star domatic number of G. $\square$

2. Basic properties of the signed star domatic number

In this section we study basic properties of $d_{SS}(G)$.

Theorem 2. *Let G be a graph of size m with signed star domination number $\gamma_{SS}(G)$ and signed star domatic number $d_{SS}(G)$. Then*

$$\gamma_{SS}(G) \cdot d_{SS}(G) \le m.$$

Proof. If $\{f_1, f_2, \dots, f_d\}$ is a signed star dominating family on G such that $d = d_{SS}(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{SS}(G) &= \sum_{i=1}^{d} \gamma_{SS}(G) \\ &\le \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) \\ &= \sum_{e \in E(G)} \sum_{i=1}^{d} f_i(e) \\ &\le \sum_{e \in E(G)} 1 \\ &= m \end{aligned}$$

as desired. $\square$

Theorem 3. *If G is a graph with minimum degree δ, then*

$$1 \le d_{SS}(G) \le \delta.$$

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a signed star dominating family on G such that $d = d_{SS}(G)$ and let $v \in V(G)$ be a vertex of minimum degree δ. Then

$$\begin{aligned} d &= \sum_{i=1}^{d} 1 \\ &\le \sum_{i=1}^{d} \sum_{e \in E(v)} f_i(e) \\ &= \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) \\ &\le \sum_{e \in E(v)} 1 \\ &= \delta. \quad \square \end{aligned}$$

An immediate consequence of Theorem 3 is:

Corollary 4. *For any tree T of order $n \ge 2$, $d_{SS}(T) = 1$.*

Theorem 5. *The signed star domatic number is an odd integer.*

Proof. Let G be an arbitrary graph, and suppose that $d = d_{SS}(G)$ is even. Let $\{f_1, f_2, \dots, f_d\}$ be the corresponding signed star dominating family on G. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \le 1$. But on the left-hand side of this inequality, a sum of an even number of odd summands occurs. Therefore it is an even number, and we obtain $\sum_{i=1}^{d} f_i(e) \le 0$ for each $e \in E(G)$. This forces

$$\begin{aligned} d &= \sum_{i=1}^{d} 1 \\ &\le \sum_{i=1}^{d} \sum_{e\in E(v)} f_i(e) \\ &= \sum_{e\in E(v)} \sum_{i=1}^{d} f_i(e) \\ &\le 0 \end{aligned}$$

which is a contradiction. □

An immediate consequence of Theorems 3 and 5 is:

Corollary 6. *For $n \ge 3$, $d_{SS}(C_n) = 1$.*

Theorem 7. *Let G be a graph, and let v be a vertex of even degree $d_G(v) = 2t$ with an integer $t \ge 1$. Then*

$$d_{SS}(G) \le \begin{cases} t & \text{when } t \text{ is odd} \\ t-1 & \text{when } t \text{ is even}. \end{cases}$$

Proof. Let $d = d_{SS}(G)$. Since $d_G(v)$ is even, we observe that

$$f(v) = \sum_{e\in E(v)} f(e) \ge 2 \tag{1}$$

for each signed star dominating function f on G. Let now $\{f_1, f_2, \ldots, f_d\}$ be a corresponding signed star dominating family on G. Using inequality (1) and $\sum_{i=1}^{d} f_i(e) \le 1$ for each edge $e \in E(G)$, we obtain

$$2t \ge \sum_{e\in E(v)} \sum_{i=1}^{d} f_i(e) = \sum_{i=1}^{d} \sum_{e\in E(v)} f_i(e) \ge \sum_{i=1}^{d} 2 = 2d.$$

This yields $d \le t$ immediately. In the case that t is even, Theorem 5 implies $d \le t - 1$, and the proof is complete. □

Restricting our attention to graphs G of even minimum degree, this theorem leads to a considerably improvement of the upper bound given in Theorem 3.

Corollary 8. *If G is a graph of even minimum degree $\delta \ge 2$, then*

$$d_{SS}(G) \le \begin{cases} \dfrac{\delta}{2} & \text{when } \delta \equiv 2 \pmod 4 \\ \dfrac{\delta-2}{2} & \text{when } \delta \equiv 0 \pmod 4. \end{cases}$$

Theorem 5 and Corollary 8 immediately lead to the next result.

Corollary 9. *If G has a vertex of degree one, two or four, then $d_{SS}(G) = 1$.*

Since every graph of odd order has a vertex of even degree, the following bounds are immediate by Theorem 7.

Corollary 10. *If G is a graph of odd order $n \ge 3$, then*

$$d_{SS}(G) \le \begin{cases} \dfrac{n-1}{2} & \text{when } n \equiv 3 \pmod 4 \\ \dfrac{n-3}{2} & \text{when } n \equiv 1 \pmod 4. \end{cases}$$

In the next section, we will see that the complete graph K_n with odd n will achieve the bound given in Corollary 10.

3. Signed star domatic number of regular graphs

In this section we determine values of the signed star domatic number for some classes of regular graphs. First we derive a structural result on $(2r + 1)$-regular graphs with maximal possible signed star domatic number.

Theorem 11. *Let G be a $(2r + 1)$-regular graph. If $d = d_{SS}(G) = 2r + 1$ and $\{f_1, f_2, \ldots, f_d\}$ is a signed domatic family on G, then $\sum_{i=1}^{d} f_i(e) = 1$ for each $e \in E(G)$, and $f_i(v) = 1$ for each $v \in V(G)$ and each $i \in \{1, 2, \ldots, 2r + 1\}$.*

Proof. Let v be an arbitrary vertex of G. Because of $\sum_{i=1}^{d} f_i(e) \le 1$ for each $e \in E(G)$, the sum contains at least r summands which have the value -1. Using the fact that $f_i(v) = \sum_{e\in E_G(v)} f_i(e) \ge 1$ for each $i \in \{1, 2, \ldots, 2r+1\}$, we observe that each of these sums contains at least $r+1$ summands which have the value 1. Consequently, the sum

$$\sum_{e\in E_G(v)} \sum_{i=1}^{d} f_i(e) = \sum_{i=1}^{d} \sum_{e\in E_G(v)} f_i(e) \tag{2}$$

contains at least dr summands of value -1 and least $d(r+1)$ summands of value 1. As the sum (2) consists of exactly $d(2r+1)$ summands, we conclude that $\sum_{i=1}^{d} f_i(e)$ contains exactly r summands of value -1 and $\sum_{e\in E_G(v)} f_i(e)$ contains exactly $r+1$ summands of value 1 for each $i \in \{1, 2, \ldots, 2r+1\}$. This leads to the desired result, and the proof is complete. $\square$

Theorem 12. *Let G be a k-regular and 1-factorable graph. Then*

$$d_{SS}(G) = \begin{cases} k & \text{when } k \text{ is odd} \\ \dfrac{k}{2} & \text{when } k \equiv 2 \pmod 4 \\ \dfrac{k-2}{2} & \text{when } k \equiv 0 \pmod 4. \end{cases}$$

Proof. Let $\{M_1, M_2, \ldots, M_k\}$ be a 1-factorization of G. We distinguish three cases.

Case 1. Assume that k is odd. For $j = 1, 2, \ldots, k$, define $f_j : E(G) \to \{-1, 1\}$ by

$$f_j(e) = \begin{cases} 1 & \text{if } e \in \bigcup_{i=2j-1}^{2j-1+\lfloor \frac{k}{2} \rfloor} M_i \\ -1 & \text{otherwise,} \end{cases}$$

where the indices are taken modulo k. Obviously $\{f_1, f_2, \ldots, f_k\}$ is a signed star dominating family on G, and the desired result follows from Theorem 3.

Case 2. Assume that $k \equiv 2 \pmod 4$. For $j = 1, 2, \ldots, \frac{k}{2}$, define $f_j : E(G) \to \{-1, 1\}$ by

$$f_j(e) = \begin{cases} -1 & \text{if } e \in \bigcup_{i=(j-1)\left(\frac{k}{2}-1\right)+1}^{j(\frac{k}{2}-1)} M_i \\ 1 & \text{otherwise} \end{cases}$$

where the indices are taken modulo k. Obviously $\{f_1, f_2 \ldots, f_{\frac{k}{2}}\}$ is a signed star dominating family on G, and the desired result follows from Theorem 7.

Case 3. Assume that $k \equiv 0 \pmod 4$. For $j = 1, 2, \ldots, \frac{k}{2} - 1$, define $f_j : E(G) \to \{-1, 1\}$ as in *Case* 2. Obviously, $\{f_1, f_2, \ldots, f_{\frac{k}{2}-1}\}$ is a signed star dominating family on G, and Theorem 7 leads to the desired result. $\square$

Theorem 13. *Let G be factorable into k Hamiltonian cycles. Then*

$$d_{SS}(G) = \begin{cases} k & \text{if } k \text{ is odd} \\ k-1 & \text{if } k \text{ is even}. \end{cases}$$

Proof. Let G be a Hamiltonian factorable graph, and let $\{C_1, C_2, \ldots, C_k\}$ be a Hamiltonian factorization of G. We distinguish two cases.

Case 1. Assume that k is odd. For $j = 1, 2, \ldots, k$, define $f_j : E(G) \to \{-1, 1\}$ by

$$f_j(e) = \begin{cases} 1 & \text{if } e \in \bigcup_{i=2j-1}^{2j-1+\lfloor \frac{k}{2} \rfloor} E(C_i) \\ -1 & \text{otherwise} \end{cases}$$

where the indices are taken modulo k. Obviously, $\{f_1, f_2, \ldots, f_k\}$ is a signed star dominating family on G.

Case 2. Assume that k is even. Let $C_k = \{e_1, e_2, \ldots, e_n\}$. We consider two subcases.

Subcase 2.1. Assume that n is odd. For $j = 1, 2, \ldots, k-1$, define $g_j : E(G) \to \{-1, 1\}$ by

$$g_j(e) = \begin{cases} 1 & \text{if } e \in \displaystyle\bigcup_{i=2j-1}^{2j-1+\lfloor\frac{k-1}{2}\rfloor} E(C_i) \\ -1 & \text{if } e \notin E(C_k) \cup \left(\displaystyle\bigcup_{i=2j-1}^{2j-1+\lfloor\frac{k-1}{2}\rfloor} E(C_i)\right) \end{cases}$$

where the indices are taken modulo $k-1$, and define $g_j(e_j) = 1$ and $g_j(e_i) = -1$ $(i \neq j)$ if i and j have the same parity and $g_j(e_i) = 1$ $(i \neq j)$ if i and j have different parity.

Obviously, $\{g_1, g_2, \ldots, g_{k-1}\}$ is a signed star dominating family on G.

Subcase 2.2. Assume that n is even. For $j = 1, 2, \ldots, k-1$, define $h_j : E(G) \to \{-1, 1\}$ by $h_j(e) = g_j(e)$ if $e \in \cup_{i=1}^{k-1} E(C_i)$ and $h_j(e_i) = (-1)^{i+j}$ for each i. Obviously $\{h_1, h_2, \ldots, h_{k-1}\}$ is a signed star dominating family on G.

Now the results follows by Theorem 7. $\square$

According to Theorems 12 and 13 and the following two well-known results, we can determine the signed star domatic number of complete graphs.

Theorem A. *The complete graph K_{2k} is 1-factorable.*

Theorem B. *For every positive integer k, the graph K_{2k+1} is Hamiltonian factorable.*

Theorem 14. *For $n \geq 3$,*

$$d_{SS}(K_n) = \begin{cases} n-1 & \text{when } n \text{ is even} \\ \dfrac{n-1}{2} & \text{when } n \equiv 3 \pmod 4 \\ \dfrac{n-3}{2} & \text{when } n \equiv 1 \pmod 4. \end{cases}$$

4. Signed star domatic number of complete bipartite graphs

In this section we determine the signed star domatic number of complete bipartite graphs. Our first lemma gives the exact value of the signed star domination number of complete bipartite graphs $K_{m,n}$. By Theorem 12 we may assume $n > m$.

Lemma 15. *For $n > m$,*

$$\gamma_{SS}(K_{m,n}) = \begin{cases} n & \text{if } m, n \text{ are both odd,} \\ 2n & \text{if } m \text{ is even,} \\ 2m & \text{if } m \text{ is odd, } n \text{ is even and } n \leq 2m, \\ n & \text{if } m \text{ is odd, } n \text{ is even and } n > 2m. \end{cases}$$

Proof. Let f be a $\gamma_{SS}(G)$-function. Since $f(v) \geq 2$ for each even vertex and $f(v) \geq 1$ for every odd vertex, we have

$$\gamma_{SS}(K_{m,n}) \geq \begin{cases} n & \text{if } m, n \text{ are both odd,} \\ 2n & \text{if } m \text{ is even,} \\ 2m & \text{if } m \text{ is odd, } n \text{ is even and } n \leq 2m, \\ n & \text{if } m \text{ is odd, } n \text{ is even and } n > 2m. \end{cases}$$

Now it suffices to define a signed star dominating function on G which attains the aforementioned bound. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be a bipartition of $V(G)$, and let $a_{i,j}$ denote the edge x_iy_j $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$. We consider the following cases. Note that in the *Cases* 1 and 3 the indices are taken modulo m, and in the *Cases* 2, 4 modulo n.

Case 1. Assume that m, n are both odd.

Define

$$g(a_{i,j}) = \begin{cases} +1 & \text{if } (i-1)\left\lceil \dfrac{n}{2} \right\rceil + 1 \leq j \leq i\left\lceil \dfrac{n}{2} \right\rceil \\ -1 & \text{otherwise.} \end{cases}$$

Case 2. Assume that m is even.

Define

$$g(a_{i,j}) = \begin{cases} +1 & \text{if } (i-1)\left(\frac{m}{2}+1\right)+1 \le j \le i\left(\frac{m}{2}+1\right) \\ -1 & \text{otherwise.} \end{cases}$$

Case 3. Assume that m is odd and n is even and $n \le 2m$.

Define

$$g(a_{i,j}) = \begin{cases} +1 & \text{if } (j-1)\left(\frac{n}{2}+1\right)+1 \le i \le j\left(\frac{n}{2}+1\right) \\ -1 & \text{otherwise.} \end{cases}$$

Case 4. Assume that m is odd and n is even and $n > 2m$. Define

$$g(a_{i,j}) = \begin{cases} +1 & \text{if } (i-1)\left\lfloor\frac{m}{2}\right\rfloor+1 \le j \le i\left\lfloor\frac{m}{2}\right\rfloor+1 \\ -1 & \text{otherwise.} \end{cases}$$

It is easy to verify that in all cases g is a signed star dominating function on G which completes the proof. □

Theorem 16. *For $n > m$,*

$$d_{SS}(K_{m,n}) = \begin{cases} m & \text{if } m, n \text{ are both odd,} \\ \frac{m}{2} & \text{if } m \text{ is even}, m \equiv 2 \pmod 4, \\ \frac{m}{2}-1 & \text{if } m \text{ is even}, m \equiv 0 \pmod 4, \\ \frac{n}{2} & \text{if } m \text{ is odd}, n \text{ is even and } n \equiv 2 \pmod 4 \text{ and } n \le 2m, \\ \frac{n}{2}-1 & \text{if } m \text{ is odd}, n \text{ is even and } n \equiv 0 \pmod 4 \text{ and } n \le 2m, \\ m & \text{if } m \text{ is odd}, n \text{ is even and } n > 2m. \end{cases}$$

Proof. By Theorems 2 and 5 and Lemma 15, the right-hand numbers are upper bounds of $d_{SS}(K_{m,n})$. It suffices to define a signed star dominating family on G which attains the aforementioned bounds. Let $f : E(G) \longrightarrow \{-1, 1\}$ be the function defined in the proof of Lemma 15. We consider the following cases: Note that in the *Cases* 1, 2, 3 and 6 the indices are taken modulo m, and in the *Cases* 4, 5 modulo n.

Case 1. Assume that m, n are both odd. For $1 \le k \le m$, define $f_k(a_{i,j}) = f(a_{i,j+(k-1)\lfloor\frac{m}{2}\rfloor})$, where $1 \le i \le m$ and $1 \le j \le n$.

Case 2. Assume that m is even and $m \equiv 2 \pmod 4$. Now for $1 \le k \le \frac{m}{2}$, define $f_k(a_{i,j}) = f(a_{i,j+(k-1)(\frac{m}{2}-1)})$, where $1 \le i \le m$ and $1 \le j \le n$.

Case 3. Assume that m is even and $m \equiv 0 \pmod 4$. Now for $1 \le k \le \frac{m}{2}-1$, define $f_k(a_{i,j}) = f(a_{i,j+(k-1)(\frac{m}{2}-1)})$, where $1 \le i \le m$ and $1 \le j \le n$.

Case 4. Assume that m is odd, n is even, $n \le 2m$ and $n \equiv 2 \pmod 4$.

For $1 \le k \le \frac{n}{2}$, define $f_k(a_{i,j}) = f(a_{i+(k-1)(\frac{m}{2}-1),j})$, where $1 \le i \le m$ and $1 \le j \le n$.

Case 5. Assume that m is odd, n is even, $n \le 2m$ and $n \equiv 0 \pmod 4$.

For $1 \le k \le \frac{n}{2}-1$, define $f_k(a_{i,j}) = f(a_{i+(k-1)(\frac{m}{2}-1),j})$, where $1 \le i \le m$ and $1 \le j \le n$.

Case 6. Assume that m is odd, n is even and $n > 2m$. Now for $1 \le k \le m$, define $f_k(a_{i,j}) = f(a_{i,j+(k-1)\lceil\frac{m}{2}\rceil})$, where $1 \le i \le m$ and $1 \le j \le n$.

In each case, $\{f_i\}$ is a signed star dominating family on G. This completes the proof. □

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